A MODEL FOR THE RELATIVELY FREE GRADED ALGEBRA OF BLOCK TRIANGULAR MATRICES WITH ENTRIES FROM A GRADED ALGEBRA

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Abstract. Let G be a group and A be a G-graded algebra satisfying a polynomial identity. We buid up a model for the relative free G-graded algebra and we obtain, as an application, the "factoring" property for the T_G-ideals of block triangular matrices with entries from the finite dimensional Grassmann algebra E for some particular $\mathbb{Z}_2$-grading.

1. introduction

Let A be an algebra over a field F. If A satisfies a polynomial identity (PI-algebra), we shall indicate by $T(A)$ the set of its polynomial identities. it is well known that if F is a field of characteristic 0, all of the polynomial identities of A come from the multilinear ones. If we set V_n to be the set of polynomials that are linear in the variables $\{x_1, \dots, x_n\}$ we can form for any $n \in \mathbb{N}$, the factor space

$$V_n(A) = V_n/(V_n \cap T(A)).$$

We shall call *n-th codimension* of A the dimension of $V_n(A)$ and we denote it by $c_n(A)$. As a vector space, $V_n \cap T(A)$ has a factorial growth while $V_n(A)$ grows at most exponentially (see the famous work of Regev [21]), then it is more useful to study the vector space $V_n(A)$. In [15] and [16] Giambruno and Zaicev proved the limit

$$\lim_{n\to+\infty} \sqrt[n]{c_n(A)}$$

always exists and is a non-negative integer, called the *PI-exponent* of A or, in symbols, $\exp(A)$. Using the language of varieties, we say that the variety generated by A is the class

$$\mathcal{V} = \mathcal{V}(A) = \{B \text{ associative algebras} \mid T(A) \subseteq T(B)\ \}.$$

We say a variety to be minimal with respect its exponent if for any proper subvariety $\mathcal{U}$ one has $\exp(\mathcal{U}) < \exp(\mathcal{V})$. We say a PI-algebra to be *minimal* if it generates a minimal variety. In [14] Giambruno and Zaicev proved that minimal varieties are determined by the T-ideals of the Grassmann envelope of the so called "minimal superalgebras". We recall that if A is a *superalgebra* (or a $\mathbb{Z}_2$-graded algebra) such that $A = A^0 \oplus A^1$, the Grassmann envelope of A is the superalgebra $G(A) = E^0 \otimes A^0 \oplus E^1 \otimes A^1$, where E is the infinite dimensional Grassmann algebra. If the ground field F is algebraically closed, such minimal superalgebras can be realized as graded subalgebras of block triangular matrix algebras endowed with a $\mathbb{Z}_2$-grading. By the Theorem of Lewin, it turns out that the T-ideals of the minimal superalgebras are products of the T-ideals corresponding to diagonal blocks, i.e., they satisfy the factoring property. These results also allow to solve into positive a conjecture of

Drensky about the factorability of T-ideals of block triangular matrices with entries from minmal algebras (see [12] and [11]).

Following this line of research, we consider a block triangular matrix

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix},$$

where A and B are PI G-graded algebras and M is an A, B-bimodule. In [9] Di Vincenzo and La Scala gave a proof for the Theorem of Lewin in the graded case. We recall that a map $|\cdot| : \{1, \dots, n\} \to G$ induces a G-grading over $M_n(F)$ such that for each i, j the homogeneous degree of the matrix unit e_{ij} is $|j||i|^{-1}$. If G is a finite abelian group, a G-grading over $M_n(F)$ is *G-regular* if $|\cdot|$ is surjective and its fiber are equipotent. Then if A and B are matrix algebras, to say $A = M_n(F)$ and $B = M_m(F)$, graded by a finite abelian group G, R has the factoring property for its ideal of graded polynomial identities if and only if one between A and B has a G-regular grading. In the proof of the Theorem of Lewin, both in the ordinary and in the graded case, one prominent role is played by the relatively free (graded) algebra. In [23] Procesi showed the k-generated relatively free algebra of $M_n(F)$ is isomorphic to the k-generated algebra of generic matrices over the polynomial ring. Moreover in [3] Berele constructed the k-generated relatively free algebras of the minimal algebras $M_n(E)$ and $M_{a,b}(E)$. It turned out they are isomorphic to some k-generated subalgebra of generic matrices over the supercommutative polynomial algebra.

In this paper we consider a PI G-graded algebra A and we construct a model for its relatively free G-graded algebra. It turns out it is isomorphic to the algebra of generic matrices with entries from $F\langle X\rangle/(F\langle X\rangle \cap T_G(A))$. We use the model to prove that if two graded algebras A and B graded by finite abelian group G have the same T_G-ideal, then

$$T_G(UT(d_1, \dots, d_m; A)) = T_G(UT(d_1, \dots, d_m; B)).$$

We also give a proof of the fact that $UT(d_1, \dots, d_m; E)$ has the factoring property when E is naturally graded. We also observed the property falls when dealing with the grading induced by the map $|\cdot|_{k^*}$ of the work of Di Vincenzo and Da Silva (see [10]). The paper is organized as follows. Sections 2 is dedicated to the main definitions concerning the graded polynomial identities. In Sections 3 we present the main tool of the paper: the graded version of the Theorem of Lewin obtained by Di Vincenzo and La Scala in [9]. In Section 4 we present the results about the $\mathbb{Z}_2$-graded identities of the infinite dimensional Grassmann algebra as well as in the paper by Di Vincenzo and Da Silva (see [10]). In Section 5 we describe the general setting of the so called G-prime algebras. As a result, we obtain that, under certain conditions, they satisfy the same graded identities of $M_n(F)$. We say that the results of Sections 4 and 5 will be used as applications of the principal Theorem. Section 6 is devoted to the construction of the model for the relatively free graded algebra of the upper traingular block matrices with entries from a PI G-graded algebra. We obtain the factoring property for $UT(d_1, \dots, d_m; A)$, where A is a G-graded algebra having the same T_G ideal of $M_n(F)$ when G-regularly graded by a finite abelian group. Moreover we obtain the factoring property for $UT(d_1, \dots, d_m; E)$ when $\mathbb{Z}_2$-graded by the $\mathbb{Z}_2$-grading induced by the natural one over E. We also mention that the factoring property for the verbally prime algebras has been completely solved by Berele and Regev in the ungraded case (see [4]).

2. Graded structures

All fields we refer to are assumed to be of characteristic zero and all algebras we consider are associative and unitary.

Let $(G,\cdot)=\{g_1,\dots,g_r\}$ be any group, and let F be a field. If A is an associative F-algebra, we say that A is a *G-graded* algebra (or G-algebra) if there are subspaces A^g for each $g\in G$ such that

$$A=\bigoplus_{g\in G}A^g \text{ and } A^gA^h\subseteq A^{gh}.$$

If $0\neq a\in A^g$ we say that a is *homogeneous of G-degree g* or *G-graded homogeneous of G-degree g*, and we write $\deg(a)=g$.

We define a free object; let $\{X^g\mid g\in G\}$ be a family of disjoint countable sets. Put $X=\bigcup_{g\in G}X^g$ and denote by $F\langle X|G\rangle$ the free associative algebra freely generated by the set X. An indeterminate $x\in X$ is said to be of *homogeneous G-degree g*, written $\|x\|=g$, if $x\in X^g$. We always write x^g if $x\in X^g$. The homogeneous G-degree of a monomial $m=x_{i_1}x_{i_2}\cdots x_{i_k}$ is defined to be $\deg(m)=\deg(x_{i_1})\cdot\deg(x_{i_2})\cdot\dots\cdot\deg(x_{i_k})$. For every $g\in G$, we denote by $F\langle X|G\rangle^g$ the subspace of $F\langle X|G\rangle$ spanned by all the monomials having homogeneous G-degree g. Notice that $F\langle X|G\rangle^gF\langle X|G\rangle^{g'}\subseteq F\langle X|G\rangle^{gg'}$ for all $g,g'\in G$. Thus

$$F\langle X|G\rangle=\bigoplus_{g\in G}F\langle X|G\rangle^g$$

proves $F\langle X|G\rangle$ to be a G-graded algebra. The elements of the G-graded algebra $F\langle X|G\rangle$ are referred to as G-graded polynomials or, simply, graded polynomials. An ideal I of $F\langle X|G\rangle$ is said to be a T_G-ideal if it is invariant under all F-endomorphisms $\varphi: F\langle X|G\rangle\to F\langle X|G\rangle$ such that $\varphi(F\langle X|G\rangle^g)\subseteq F\langle X|G\rangle^g$ for all $g\in G$. If A is a G-graded algebra, a G-graded polynomial $f(x_1,\dots,x_n)$ is said to be a *graded polynomial identity* of A if $f(a_1,a_2,\cdots,a_t)=0$ for all $a_1,a_2,\cdots,a_t\in\bigcup_{g\in G}A^g$ such that $a_k\in A^{\deg(x_k)}$, $k=1,\cdots,t$. We denote by $T_G(A)$ the ideal of all graded polynomial identities of A. It is a T_G-ideal of $F\langle X|G\rangle$. If A is ungraded, i.e., graded by the trivial group, we speak about polynomial identities and T-ideal of A. We recall that if the group G is finite and A is a G-graded PI-algebra, then it satisfies a polynomial identity (see [1], [5]). Moreover, we recall that if two G-algebras A and B satisfy the same graded identities, i.e., $T_G(A)=T_G(B)$, then they satisfy the same identities, i.e., $T(A)=T(B)$.

One of the fundamental tool in what follows is the relatively free graded algebra. We recall its definition. We shall denote by the symbol $U_{G,k}(A)$ the relatively free algebra

$$F\langle x_1^{g_1},\dots,x_k^{g_1},\dots,x_1^{g_r},\dots,x_k^{g_r},\dots\rangle/(F\langle x_1^{g_1},\dots,x_k^{g_1},\dots,x_1^{g_r},\dots,x_k^{g_r},\dots\rangle\cap T_G(A))$$

and we shall call it the *relatively free G-algebra of A in k graded variables.*

Furthermore, we give some basic definition about graded related structures. An ideal I is called G-graded if $I=\bigoplus_{g\in G}(I\cap A^g)$. For any ideal I of A we shall denote by I_{gr} the largest graded ideal of A contained in I. We shall refer to I as a G-ideal, too. We say that a G-algebra is G-simple if it has no proper graded ideals. Moreover, we shall denote by $Z_{gr}(A)$ the maximal graded subalgebra in the center of A $(Z(A))$, i.e., the *graded center* of A.

3. $\mathbb{Z}_2$-graded identities for the Grassmann algebra

In this section we recall the main tools and definitions necessary for the study of graded polynomial identities for the Grassmann algebra that we shall indicate by E.

The algebra E can be constructed as follows. Let $F\langle X\rangle$ be the free algebra of countable rank on $X = \{x_1, x_2, \ldots\}$. If I is the two-sided ideal of $F\langle X\rangle$ generated by the set of polynomials $\{x_ix_j + x_jx_i | i, j \geq 1\}$, then $E = F\langle X\rangle/I$. If we write $e_i = x_i + I$ for $i = 1, 2, \ldots$, then E has the following presentation:

$$E = \langle 1, e_1, e_2, \ldots | e_ie_j = -e_je_i, \text{for all } i, j \geq 1\rangle.$$

We say that the vector space V generated by X over F is the generating vector space for E. Moreover, the set

$$B = \{1, e_{i_1} \cdots e_{i_k} | 1 \leq i_1 < \cdots < i_k\}$$

is a basis of E over F. Sometimes it is convenient to write E in the form $E = E^{(0)} \oplus E^{(1)}$, where

$$E^0 := \text{span}\{1, e_{i_1} \cdots e_{i_{2k}} | 1 \leq i_1 < \cdots < i_{2k}, k \geq 0\},$$

$$E^1 := \text{span}\{1, e_{i_1} \cdots e_{i_{2k+1}} | 1 \leq i_1 < \cdots < i_{2k+1}, k \geq 0\}.$$

It is easily checked that the decomposition $E = E^0 \oplus E^1$ is a $\mathbb{Z}_2$-grading of E called the natural grading. Notice that E^0 coincides with the center of E. We give a look at the whole class of homogeneous $\mathbb{Z}_2$-grading of E. For more details we refer to the work of Di Vincenzo and Da Silva ([10]).

For a homogeneous $\mathbb{Z}_2$-grading of E we mean any $\mathbb{Z}_2$-grading such that the generating vector space V is a homogeneous subspace. This is equivalent to consider a map

$$\varphi : V \to \mathbb{Z}_2.$$

If $w = e_{i_1}e_{i_2} \cdots e_{in} \in E$ then the set $\text{Supp}(w) := \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ is the support of w and we define the $\mathbb{Z}_2$-grading of w by

$$\deg(e_{i_1}e_{i_2} \cdots e_{i_n}) = \deg(e_{i_1}) + \cdots + \deg(e_{i_n}).$$

If, for all $e_i \in B$, one has $\deg(e_i) = 1 \in \mathbb{Z}_2$, then we obtain the natural $\mathbb{Z}_2$-grading on E.

In this case, let E^0 be the homogeneous component of $\mathbb{Z}_2$-degree 0 and let E^1 be the component of degree 1. As we said above, $E^0 = Z(E)$ is the center of E and $ab + ba = 0$ for all $a, b \in E^1$. This means that E satisfies the following graded polynomial identities: $[y_1, y_2]$, $[y_1, z_1]$, $z_1z_2 + z_2z_1$. Now, let us consider the $\mathbb{Z}_2$-gradings on E induced by the maps $\deg(\cdot)_{k*}$, $\deg(\cdot)_\infty$, and $\deg(\cdot)_k$, defined respectively by:

$$\|e_i\|_{k*} = \begin{cases} 1 \text{ for } i = 1, \ldots, k \\ 0 \text{ otherwise,} \end{cases}$$

$$\|e_i\|_\infty = \begin{cases} 1 \text{ for } i \text{ odd} \\ 0 \text{ otherwise,} \end{cases}$$

$$\|e_i\|_k = \begin{cases} 0 \text{ for } i = 1, \ldots, k \\ 1 \text{ otherwise.} \end{cases}$$

By [10], we have the following result.

Theorem 1. *Let Y be a countable set of indeterminates of degree 0 and Z be a countable set of indeterminates of degree 1 and put $X = Y \cup Z$. Then:*

(1) *The $T_{\mathbb{Z}_2}$-ideal of E graded by $\deg(\cdot)_\infty$ is generated by the polynomial*
$$[x_1, x_2, x_3].$$
(2) *The $T_{\mathbb{Z}_2}$-ideal of E graded by $\deg(\cdot)_{k^*}$ is generated by the polynomials*
$$[x_1, x_2, x_3], z_1 z_2 \cdots z_{k+1}.$$
(3) *The $T_{\mathbb{Z}_2}$-ideal of E graded by $\deg(\cdot)_k$ is generated by the polynomials*
$$[x_1, x_2, x_3], [y_1, y_2] \cdots [y_{k-1}, y_k][y_{k+1}, x] \text{ (if } k \text{ is even)},$$
$$[y_1, y_2] \cdots [y_k, y_{k+1}] \text{ (if } k \text{ is odd)},$$
$$g_{k-l+2}(z_1, \dots, z_{k-l+2})[y_1, y_2] \cdots [y_{l-1}, y_l] \text{ (if } l \le k),$$
$$[g_{k-l+2}(z_1, \dots, z_{k-l+2}), y_1][y_2, y_3] \cdots [y_{l-1}, y_l] \text{ (if } l \le k,\ l \text{ is odd)},$$
$$g_{k-l+2}(z_1, \dots, z_{k-l+2})[z, y_1][y_2, y_3] \cdots [y_{l-1}, y_l] \text{ (if } l \le k,\ l \text{ is odd)}.$$

Remark 2. *We note that the Grassmann algebra satisfies a graded monomial identity only if its $\mathbb{Z}_2$-grading is induced by $\deg(\cdot)_{k^*}$.*

Remark 3. *It is easy to be seen that a basis for the relatively free $\mathbb{Z}_2$-graded algebra of E with the grading induced by $\deg(\cdot)_\infty$ is the following:*
$$y_{i_1}^{n_1} \cdots y_{i_l}^{(n_l)} z_{j_1} \cdots z_{j_m},$$
where $i_1 < \cdots < i_l$, $n_1, \dots, n_l \in \mathbb{N}$ $j_1 < \cdots < j_m$.

On the other side, a basis for the relatively free $\mathbb{Z}_2$-graded algebra of E with the grading induced by $\deg(\cdot)_{k^*}$ is the following:
$$y_{i_1} \cdots y_{i_l} z_{j_1} \cdots z_{j_m},$$
where $i_1 \le \cdots \le i_l$, $j_1 < \cdots < j_m$ and $m \le k-1$.

4. The graded Thorem of Lewin

We resume the work of Di Vincenzo and La Scala (see [9]) for the generalization of the Theorem of Lewin (see [19]) at the graded case. In what follows the grading group is supposed to be finite.

Ler A, B be G-graded algebras and M be an A,B-bimodule, then it is possible to consider the G-algebra
$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}.$$
The following result holds.

Proposition 4. *Let A and B be PI G-algebras. If the M contains a countable free set $\{ui\}$ of homogeneous elements such that $\deg(x_i) = \deg(u_i)$ for any $i \ge 1$, then $T_G(R) = T_G(A)T_G(B)$.*

We fix a map $|\cdot| : 1, 2, \cdots, n \to G$. Then $|\cdot|$ induces a grading on $M_n(F)$ by setting $|\deg(e_{ij})| = |j||i|^{-1}$, for all matrix units e_{ij}. Indeed this is an elementary grading defined by $(|1|, \dots, |n|)$. If we specialize the algebra A with a matrix algebras, to say $A \subseteq M_n(F)$, with an elementary G-grading, the authors introduce the notion of G-regular grading. In the case G is abelian, we have the following equivalence result.

Theorem 5. *The G-algebra A is G-regular if and only if the map $|\cdot|$ is surjective and all its fibers are equipotent.*

Note that the G-regularity of A is verified in particular when the order of G is exactly n and the map $|\cdot|$ is bijective. This is the case, for instance, when we consider the Vasilovsky $\mathbb{Z}_n$-grading of $M_n(F)$ (see [8] and [24]). Moreover, for the ordinary case, that is for $G = \{1_G\}$, the algebra A is regular. We close the section with the main result of [9].

Theorem 6. *Let R be the G-graded block-triangular matrix algebra defined as above, where $A \subseteq M_n(F)$, $B \subseteq M_m(F)$ are G-algebras and M an A,B-bimodule. If one between A and B is G-regular, then the T_G-ideal $T_G(R)$ factorizes as: $T_G(R) = T_G(A)T_G(B)$.*

Corollary 7. *Let*

$$R = \begin{pmatrix} A_{11} & A_{12} & \dots & \dots & A_{1n} \\ 0 & A_{21} & \dots & \dots & A_{2n} \\ \vdots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & A_{nn} \end{pmatrix}$$

be a G-algebra, where the A_{ij}'s are G- graded subalgebras of some G-graded matrix algebra. Suppose that for any $i \leq n$ the G-grading over A_{ii} is G-regular. Then

$$T_G(R) = T_G(A_{11})T_G(A_{22}) \cdots T_G(A_{nn}).$$

Proof. We prove the statement by induction on n. If $n = 2$, we are in the hypothesis of Theorem 6 and we are done. Suppose true the assertion for $n-1$, where $n \geq 3$. We consider

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & \dots & A_{1n-1} \\ 0 & A_{21} & \dots & \dots & A_{2n-1} \\ \vdots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & A_{n-1n-1} \end{pmatrix},$$

then

$$R = \begin{pmatrix} A & M \\ 0 & A_{nn} \end{pmatrix}.$$

Due to the fact that R is a G-algebra, we have A is a G-algebra, too. Now we are again in the hypothesis of Theorem 6 and the proof follows. $\square$

5. Central graded prime algebras

In this section, we want to focus on graded prime central algebras and we give the proof of the fact that, under the hypothesis over G to be ordered or finite abelian, their T_G-ideals are the same of those of matrices over a field equipped with an elementary G-grading. For this purpose, we shall introduce some definitions and results obtained by Balaba in [2]. We start off with the following definition.

Definition 8. *A G-ideal P of a G-algebra A is called graded prime or G-prime if whenever $aAb \subseteq P$, where $a, b \in h(A)$, either $a \in P$ or $b \in P$. Moreover, a graded algebra A is called graded prime if (0) is a graded prime ideal of A.*

Definition 9. *We say that a G-algebra is a G-division algebra if each homogeneous element has an inverse.*

It is easy to see that each G-division algebra is a G-prime algebra and a G-domain as well.

Definition 10. *A group $(G, \cdot)$ is said to be ordered (or partially ordered) if it is equipped with a partial order $\leq$ such that for all $a, b, g \in G$, if $a \leq b$, one has that $ag \leq bg$ and $ga \leq gb$. The ordered group G is said to be totally ordered if $\leq$ is an order relation and is total, i.e., for any $a, b \in G$ $a \leq b$ or $b \leq a$.*

Before giving another example of G-prime algebra, we recall the construction of generic matrix algebras for one of the verbally prime algebras with the grading of Di Vincenzo (see [8]) and Vasilovsky ([24]); these algebras play a foundamental role in the work of Kemer about the structure of varieties of algebras (see [18]). Generally speaking, the target of the construction of generic verbally prime algebras is to find a more simple form of the relatively free algebra of any G-algebra. It turns out that the latter are representable (embeddable in a matrix algebra $M_k(B)$, where B is a commutative algebra). We follow the construction of Berele in [3]. Let X and Y be two countable sets of variables and we start off with the free associative algebra $F\langle X \cup Y\rangle$, then we introduce a $\mathbb{Z}_2$-grading by setting $\|x\| = 0$ and $\|y\| = 1$ for all $x \in X$ and $y \in Y$. Furthermore, we say for all $\mathbb{Z}_2$-homogeneous elements a,b $ab = (-1)^{\|a\|\|b\|}ba$. We denote this algebra by $F[X;Y]$ that is called *free supercommutative algebra.*

Let $k, n \in \mathbb{N}$ and a, b such that $a + b = n$ and set $X = \{x_{ij}^{(r)} | i \leq j, i, j = 1, \ldots, n, r = 1, \ldots, k\}$ and $Y = \{y_{ij}^{(r)} | i \leq j, i, j = 1, \ldots, n, r = 1, \ldots, k\}$. Then we shall construct $U_{\mathbb{Z}_n,k}(M_n(F))$ as *generic algebras*, i.e., as algebras with entries from $M_n(F[X;Y])$. For every $r = 1, \ldots, k$, let us consider the following:

$$A_r^{(i)} = \sum_{\|e_{pq}\|=(i)} x_{pq}^{(r)} e_{pq}.$$

We shall denote by $F_k(A)$ the F-algebra generated by

$$A_1^{(0)}, \ldots, A_k^{(0)}, \ldots, A_1^{(n-1)}, \ldots, A_k^{(n-1)}.$$

We have the following result.

Theorem 11. *For every $n, k \in \mathbb{N}$ $U_{\mathbb{Z}_n,k}(M_n(F)) \cong F_k(A)$.*

Now we give another example of graded prime algebra (see [7]).

Example 12. *For any $k \geq 1$, the algebra $F_k(A)$ is a $\mathbb{Z}_n$-prime algebra.*

Definition 13. *We say that an algebra A is central if $Z(A) = F$.*

We are going to focus on central G-prime PI algebras.

Example 14. *Let $A = M_n(F)$ trivially graded by a group G. Then A is a central PI-algebra because $Z(A) = F$. Moreover it is G-prime because $M_n(F)$ is a prime algebra.*

Example 15. *Let $A = F$ be a field graded by a group G. Then A is a central PI-algebra because $Z(A) = F$. Moreover it is G-prime because F is a prime algebra.*

Example 16. *Let D be a division algebra, then we consider $A = M_n(\mathbb{H})$, with the $\mathbb{Z}_n$-grading of Vasilovsky. Then A is a $\mathbb{Z}_n$-prime algebra.*

Recall that by Proposition 1 of [2], the localization A_S of A over S, where S is a set of homogeneous elements of the center $Z(A)$ of A, is a G-graded PI-algebra of central quotients of A. An algebra $Q(A) \supseteq A$ is called the *left (right) graded algebra of quotients* of A if:

(1) each homogeneous regular element from A is invertible in $Q(A)$;
(2) each homogeneous element $x \in Q(A)$ has the form $a^{-1}b$ (ba^{-1}), where $a, b \in h(A)$ and a is a regular element.

The next theorems are the main tools for what will follow (see [2]):

Theorem 17. *Let A be a G-prime PI-algebra, $Z(A)$ the center of A and S the set of homogeneous regular elements of $Z(A)$. Then:*

(1) $S = h(Z(A))$;
(2) *the algebra of quotients A_S is a G-graded graded prime PI-algebra;*
(3) $Z_{gr}(A_S) = Z_{gr}(A)_S$.

Theorem 18. *Let A be a G-prime PI-algebra and A_0 the algebra of central quotients of A. Then:*

(1) *A_0 is finite dimensional graded-simple over its graded center Z and Z is the graded field of quotients of $Z_{gr}(A)$;*
(2) *A_0 is the graded algebra of quotients of A;*
(3) *A and A_0 satisfy the same identities.*

Let A be a PI G-prime algebra, with center $Z(A)$. If A is central, $Z(A)$ is a field. In the case G is an abelian or an ordered group, $Z(A) = Z_{gr}(A)$ (see Lemma C.1.5.4 of [20]). By Theorems 17 and 18, A_0 is PI G-prime with center $Z_{gr}(A_0) = Z_{gr}(A)_0 = Z(A)_0 = Z(A)$ because $Z(A)$ is a field. Hence A_0 is a central PI G-prime G-simple algebra. We consider the following result given in [2].

Theorem 19. *Let A be a G-primitive graded algebra. If A is finite dimensional, then A is isomorphic to $M_n(D)(g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in G$ and a graded division ring D. Otherwise for each positive integer m there exists a graded subring S_m in R, which maps homomorphically onto $M_n(D)(g_1, \ldots, g_n)$.*

Notice that in the previous theorem, the graded domain D is merely trivially graded, then it is a division ring.

We also recall the following, well known lemma (see, for example, the book of Herstein [17]).

Lemma 20. *Let D be a domain over the algebraically closed field F such that D is algebraic over F, then $D = F$.*

Now we have the following result.

Lemma 21. *If A is a graded central simple algebra over the graded field F and B is a graded simple algebra containing F in its graded center, then $A \bigotimes_F B$ is a graded central simple algebra over F.*

Theorem 22. *Let G be an ordered or abelian group and A be a central PI G-prime algebra of dimension n^2 over its center, then A satisfies the same graded identities of $M_n(F)$ for some elementary G-grading.*

Proof. Let $\overline{F}$ be the algebraic closure of F, then we consider the algebra $A_0 \otimes_F \overline{F}$. We observe that $\overline{F}$ is G-simple, then by Lemma 21 $A_0 \otimes_F \overline{F}$ is G-simple and finite dimensional over $\overline{F}$. The algebra $A_0 \otimes_F \overline{F}$ is also G-primitive, then by Theorem 2 of [2] and by the proof of the graded version of the Theorem of Kaplansky (see [6]), it is isomorphic to $M_k(D)(g_1, \ldots, g_n)$, where D is a G-division algebra, finite dimensional over $\overline{F}$, and $g_1, \ldots, g_n \in G$. By Lemma 20, we have $D = \overline{F}$. Simple linear algebra computations show that $k = n$. Now we consider the following chain of G-isomorphisms:

$$A_0 \otimes_F \overline{F} \cong_{gr} M_n(\overline{F}) \cong_{gr} M_n(F) \otimes_F \overline{F}.$$

It turns out that $A_0 \otimes_F \overline{F}$ and $M_n(F) \otimes_F \overline{F}$ satisfy the same graded identities and, due to the fact that $\overline{F}$ is a commutative algebra, it follows that $T_G(A_0) = T_G(M_n(F))$ and the proof follows. □

Theorem 23. *Let G be an ordered or an abelian group and A be a central PI G-prime algebra over a field F. Then it satisfies the same graded identities of $M_n(F)$ endowed with some elementary grading.*

Proof. We observe that A_0 is central PI G-prime, G-simple. By Theorem 4 of [2] (see also [6]), we have that A_0 is finite dimensional over its center with dimension n^2 for some $n \in \mathbb{N}$. Now we use Theorem 22 and we are done. □

6. A model for the relatively free graded algebra of block triangular matrices

In this section we build up a model for any relatively free graded algebra. We recall that in the finite dimensional case there is a standard way to build up such a model. For more details we refer to the book of Rowen [22].

Let A be a PI G-algebra over a field F. If $d_1, d_2, \ldots, d_m$ are positive integers, we denote by $UT(d_1, \ldots, d_m; A)$ the subalgebra of the matrix algebra $M_{d_1+\cdots+d_m}(A)$ consisting of matrices of the type

$$\begin{pmatrix} A_{11} & A_{12} & \ldots & \ldots & A_{1n} \\ 0 & A_{21} & \ldots & \ldots & A_{2n} \\ \vdots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & A_{nn} \end{pmatrix},$$

where $A_{ij} \in M_{d_i \times d_j}(A)$ for each i, j. One such algebra is called the algebra of *block-triangular matrices* of size $d_1, \ldots, d_m$ over A. In what follows we are going to build up a model for the relatively free graded algebra of $UT(d_1, \ldots, d_m; A)$, where A is any PI G-algebra.

We shall use the following notation: if $f(x_1, \ldots, x_n)$ is a graded polynomial of $F\langle X|G\rangle$, we shall indicate by $\overline{x}$, the string of the homogenous indeterminates appearing in f, i.e., $\{x_1, \ldots, x_n\}$, and we shall write $f(\overline{x})$ instead of $f(x_1, \ldots, x_n)$. Moreover, if we are dealing with any graded substitution of the type $x_1^{\deg x_1} \mapsto a_1^{\deg x_1}$, we shall indicate the valuation of f by $f(\overline{a})$.

The model is based on the following construction. For each $k \in \mathbb{N}$ and $g \in G$, we define the matrix $\xi_k^{(g)} \in UT(d_1, \dots, d_m; U_G(A))$ by

$$\xi_k^{(g)} = \begin{pmatrix} B^{(g)}_{d_1\times d_1,k} & B^{(g)}_{d_1\times d_2,k} & \cdots & B^{(g)}_{d_1\times d_m,k} \\ 0 & B^{(g)}_{d_2\times d_2,k} & \cdots & B^{(g)}_{d_2\times d_m,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(g)}_{d_m\times d_m,k} \end{pmatrix}$$

Here, each $B^{(g)}_{d_r\times d_s,k}$ is a $d_r \times d_s$ matrix whose entry (i,j) is $x^{(g)}_{ij,k} + T_G(A) \in U^G(A)$, with $d_1 + \cdots + d_{r-1} + 1 \leq i \leq d_1 + \cdots + d_r$ and $d_1 + \cdots + d_{s-1} + 1 \leq j \leq d_1 + \cdots + d_s$.

We denote by $\mathcal{U}(d_1, \dots, d_m; A)$ the subalgebra of $UT(d_1, \dots, d_m; U_G(A))$, generated by the matrices $\xi_k^{(g)}$, $k \in \mathbb{N}, g \in G$, defined above (we omit an index G in order to simplify notation).

Lemma 24. *The algebra $\mathcal{U}(d_1, \dots, d_m; A)$ is a generic model for the relatively free graded algebra of $UT(d_1, \dots, d_m; A)$, i.e.,*

$$\mathcal{U}(d_1, \dots, d_m; A) \cong \frac{F\langle X|G\rangle}{T_G(UT(d_1, \dots, d_m; A))}.$$

Proof. Let $X = \bigcup_{g\in G} X^g$, where the union is disjoint and each X^g is a countable set of homogeneous indeterminates. Define the homomorphism

$$\begin{array}{rccc} \varphi: & F\langle X|G\rangle & \longrightarrow & \mathcal{U}(d_1, \dots, d_m; A) \\ & x_k^{(g)} & \mapsto & \xi_k^{(g)}. \end{array}$$

Of course φ is a graded homomorphism onto $\mathcal{U}(d_1, \dots, d_m; A)$. We shall show

$$\ker\varphi = T_G(UT(d_1, \dots, d_m; A)).$$

First, observe that $\ker\varphi \subseteq T_G(UT(d_1, \dots, d_m; A))$. Indeed, if $f = f(\overline{x}) \in \ker\varphi$, then $\varphi(f) = f(\overline{\xi}) = 0$. Since $U_G(A)$ is the relatively free algebra of A, we have that for any $\{a^{(g)}_{ij,k}\} \subseteq A$, there exists a graded homomorphism $U_G(A) \longrightarrow A$, such that $x^{(g)}_{ij} + T_G(A) \mapsto a^{(g)}_{ij,k}$. Since $f(\overline{\xi}) = 0$ in $\mathcal{U}(d_1, \dots, d_m; A)$, the image of any entry of this matrix under any homomorphisms $U_G(A) \longrightarrow A$ is zero. Then $f \in T_G(UT(d_1, \dots, d_m; A))$.

In order to show the reverse inclusion, we take $f(\overline{x}) \in F\langle X|G\rangle$ a graded polynomial identity for $UT(d_1, \dots, d_m; A)$. Let us now consider the matrix $M = f(\overline{\xi})$. Each entry (r,s) of M has the form $m_{rs}(x^{(g)}_{ij,k}) + T_G(A)$, for some polynomials $m_{rs} \in F\langle X|G\rangle$, with $1 \leq r,s \leq d_1 + \cdots + d_m$. We claim that $M = 0$. Indeed, since $f(\overline{u}) = 0$, for any $\overline{u} \in UT(d_1, \dots, d_m; A)$, we have that $m_{rs}(a^{(g)}_{ij,k}) = 0$, for any $a^{(g)}_{ij,k} \in A$ and this shows that $m_{rs} \in T_G(A)$, for any r and s, and hence $M = 0$, which concludes the lemma. $\square$

Lemma 25. *Let A and B be PI G-algebras such that $T_G(A) = T_G(B)$. Then*

$$T_G(UT(d_1, \dots, d_m; A)) = T_G(UT(d_1, \dots, d_m; B)).$$

As a consequence, for $n \in \mathbb{N}$, we have $T_G(M_n(A)) = T_G(M_n(B))$.

Proof. Let $\xi_k^{(g)}$ be the generators of $\mathcal{U}(d_1, \dots, d_m; A)$ as in the previous theorem, and η_k be the generators of $\mathcal{U}(d_1, \dots, d_m; B)$. Let

$$f(\overline{x}) \in T_G(UT(d_1, \dots, d_m; A)),$$

then we claim that $f \in T_G(UT(d_1,\dots,d_m;B))$. Let $m_{rs}(x_{ij,k}^{(g)}+T_G(A))$ be the entry (r,s) of the matrix $f(\overline{\xi})$, where m_{rs} are polynomials in the variables $x_{ij,k}^{(g)}$. Since $f \in T_G(UT(d_1,\dots,d_n))$, by Lemma 24 we have $f(\overline{\xi})=0$, then $m_{rs}(x_{ij,k}^{(g)}+T_G(A))=0$, i.e., $m_{rs}\in T_G(A)=T_G(B)$. Hence, $m_{rs}(x_{ij,k}^{(g)}+T_G(B))=0$, for every (r,s). Then, we have $f(\overline{\eta})=(m_{rs}(x_{ij,k}^{(g)}+T_G(B)))=0$, and $f\in T_G(UT(d_1,\dots,d_m;B))$. The other inclusion is analogous. □

Theorem 26. *Let G be a finite abelian group and A be a PI G-algebra such that $T_G(A)=T_G(M_n(F))$, where the grading over $M_n(F)$ is G-regular. Then for any set of positive integers $\{d_1,\dots,d_m\}$, we have*

$$T_G(UT(d_1,\dots,d_m;A))=T_G(M_{d_1}(A))T_G(M_{d_2}(A))\cdots T_G(M_{d_m}(A)).$$

Proof. If $T_G(A)=T_G(M_n(F))$, by Lemma 24 we have

$$T_G(UT(d_1,\dots,d_m;A))=T_G(UT(d_1,\dots,d_m;M_n(F))).$$

In light of the fact that the grading on $M_n(F)$ is G-regular, the grading is induced by the n-tuple $\overline{g}=(g_1,\dots,g_n)$, with equipotent fibers. It is easy to see that $M_k(M_n(F))$ is G-graded isomorphic to $M_{nk}(F)$, for any k and n, where the G-grading is induced by the kn-tuple

$$\overline{g_1}=(g_1,\dots,g_n,g_1,\dots,g_n,\dots,g_1,\dots,g_n).$$

Hence

$$T_G(UT(d_1,\dots,d_m;M_n(F)))=T_G(UT(nd_1,\dots,nd_m;F))).$$

Indeed the G-grading over each of the $M_{nd_i}(F)$'s is G-regular, then by Corollary 7, the right-hand site of the above equation is equal to

$$T_G(M_{nd_1}(F))\cdots T(M_{nd_m}(F)).$$

By Lemma 25, for each i, we have

$$T_G(M_{nd_i}(F))=T_G(M_{d_i}(M_n(F)))=T_G(M_{d_i}(A)),$$

and the result follows. □

Corollary 27. *Let G be a finite abelian group and A be a central G-prime algebra over a field F such that A_0 satisfies the same graded identities of $M_n(F)$ graded by a G-regular grading. Then for any set of positive integers $\{d_1,\dots,d_m\}$, we have*

$$T_G(UT(d_1,\dots,d_m;A))=T_G(M_{d_1}(A))T_G(M_{d_2}(A))\cdots T_G(M_{d_m}(A)).$$

Proof. By Theorem 23 we have $T_G(A)=T_G(M_n(F))$, then we use Theorem 26 and we are done. □

7. Block triangular matrices with entries from the Grassmann algebra

In this section, we shall use the model we built up in the previous section in order to obtain a "splitting" theorem for block triangular matrices with entries from the Grassmann algebra E, $\mathbb{Z}_2$-graded by the grading inherited by the natural $\mathbb{Z}_2$-grading of E.

Let A be a PI G-algebra and consider the automorphism φ of $U_G(A)$ defined by $\varphi(x_{ij,k}^{(g)})=x_{i+1j+1,k}^{(g)}$, where the sum on the indexes is taken modulo n. I.e.,

we consider the automorphism of $U_G(A)$, which sends each entry (i, j) of a generic matrix to the next variable in its diagonal.

The next lemma shows how the entries of the elements of $\mathcal{U}(n; A)$ behave with respect to this automorphism.

Lemma 28. *Let A be a PI G-algebra and $M = (m_{ij}) \in \mathcal{U}(n; A)$. Then, for each i and j, we have $\varphi(m_{ij}) = m_{i+1j+1}$.*

Proof. We use the same notations of Lemma 24. It is enough to prove it for monomials and we do it by induction on the degree of such monomial. Let $f = f(B_1, \ldots, B_r)$, where the B_t are the generic matrices of $U_G(M_n(A))$, be one such monomial. If f has degree one, then f is one of the generic matrices of $M_n(A)$ and the result holds. Suppose the result true for monomials of degree strictly less than m. If f has degree m, we write $f = gB_s$, with $g = g(B_1, \ldots, b_r) = (p_{ij})$, where g has degree $m - 1$. If $B_s = (x^{(g)}_{ij,k})$, we have $f = \left(\sum_{t=1}^{n} p_{it} x^{(g)}_{tj,k}\right)$. The (i, j) entry of f is $\sum_{t=1}^{n} p_{it} x^{(g)}_{tj,k}$. Hence $\varphi(\sum_{t=1}^{n} p_{it} x^{(g)}_{tj,k}) = \sum_{t=1}^{n} \varphi(p_{it})\varphi(x^{(k)}_{tj}) = \sum_{t=1}^{n} p_{i+1t+1} x^{(g)}_{t+1j+1,k}$ that is the $(i + 1, j + 1)$ entry of f, and we are done. $\square$

As a consequence, the next result shows that in order to verify that elements of $\mathcal{U}_G(n; A)$ are linearly independent, it is enough to verify it for their first columns.

Corollary 29. *Let A be a PI G-algebra, $\mathcal{U}(n; A)$ be the graded generic algebra of $M_n(A)$ and $\{f_1, \ldots f_n\} \subseteq \mathcal{U}(n; A)$. If f^k_i is the k-th column of f_i then set of column vectors with entries from $U_G(A)$, $\{f^k_1, \ldots, f^k_n\}$ is linearly independent over F if and only if $\{f_1, \ldots f_n\}$ is linearly independent over F.*

Theorem 30. *Let E be the Grassmann algebra endowed with the natural $\mathbb{Z}_2$-grading. If $d_1, \ldots, d_m$ are positive integers, then*
$$T_{\mathbb{Z}_2}(UT(d_1, \ldots, d_m; E)) = T_{\mathbb{Z}_2}(M_{d_1}(E)) \cdots T_{\mathbb{Z}_2}(M_{d_m}(E)).$$

Proof. We prove the theorem by induction on m. If $m = 1$ the result is obvious. We prove the result for m supposing it holds for $m - 1$. We use the same notation of the proof of Lemma 24. By Lemma 24 we have
$$T_{\mathbb{Z}_2}(UT(d_1, \ldots, d_m; E)) = T_{\mathbb{Z}_2}(\mathcal{U}(d_1, \ldots, d_m; E)).$$
The algebra $\mathcal{U}(d_1, \ldots, d_m; E)$ is generated by the matrices
$$\xi^{(g)}_k = \begin{pmatrix} B^{(g)}_{d_1\times d_1,k} & B^{(g)}_{d_1\times d_2,k} & \cdots & B^{(g)}_{d_1\times d_m,k} \\ 0 & B^{(g)}_{d_2\times d_2,k} & \cdots & B^{(g)}_{d_2\times d_m,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(g)}_{d_m\times d_m,k} \end{pmatrix}, \quad k \geq 1,$$
where g ranges over $\mathbb{Z}_2$. We consider now the algebra A generated by the matrices
$$\omega^{(g)}_k = \begin{pmatrix} B^{(g)}_{d_1\times d_1,k} & B^{(g)}_{d_1\times d_2,k} & \cdots & B^{(g)}_{d_1\times d_{m-1},k} \\ 0 & B^{(g)}_{d_2\times d_2,k} & \cdots & B^{(g)}_{d_2\times d_{m-1},k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(g)}_{d_{m-1}\times d_{m-1},k} \end{pmatrix}, \quad k \geq 1,$$
and the algebra B generated by the matrices $\eta^{(g)}_k = B^{(g)}_{d_m\times d_m,k}$, for $k \geq 1$ and $g \in \mathbb{Z}_2$.

We recall that by Lemma 24, the algebra A is a relatively free $\mathbb{Z}_2$-graded algebra of $UT(d_1, \ldots, d_{m-1}; E)$ freely generated by the matrices $\omega_k^{(g)}$ and B is a relatively free algebra of $M_{d_m}(E)$, freely generated by the matrices $\eta_k^{(g)}$.

Let M be the (A, B)-bimodule generated by the $(d_1 + \cdots + d_{m-1}) \times d_1$ matrices

$$\mu_k^{(g)} = \begin{pmatrix} B^{(g)}_{d_1 \times d_m, k} \\ B^{(g)}_{d_2 \times d_m, k} \\ \vdots \\ B^{(g)}_{d_{m-1} \times d_m, k} \end{pmatrix}, \quad k \geq 1,$$

with action given by the usual product of matrices.

By Proposition 4 (the graded theorem of Lewin), if M is freely generated by the $\mu_k^{(g)}$, we have

$$T_{\mathbb{Z}_2}(\mathcal{U}(d_1, \ldots, d_m; E)) = T_{\mathbb{Z}_2}(\mathcal{U}(d_1, \ldots, d_{m-1}; E))T_{\mathbb{Z}_2}(\mathcal{U}(d_m; E))$$

and the proof follows by induction.

In order to show that the bimodule M is freely generated by the $\mu_k^{(g)}$, we suppose the set $\{f_1, \ldots, f_n\} \subseteq B$ to be linearly independent over F and $\{h_{ij}\} \subseteq A$. We need to show that if $\sum_{i,j} h_{ij}\mu_j f_i = 0$, then each h_{ij} is zero in A. Observe that since each $\mu_k^{(g)}$ depends on disjoint sets of variables, it is enough to prove it for only one of them, say $\mu_1^{(1)} = \mu$, i.e., we need to prove that if $\{f_1, \ldots, f_n\}$ is linearly independent over F and $\{h_1, \ldots, h_n\} \subseteq A$ is such that $\sum_{i=1}^n h_i \mu f_i = 0$, then each h_i is zero.

We denote $d = d_1 + \cdots + d_{m-1}$ and rename the variables $x_{ij,k}^{(g)}$, $1 \leq i \leq d$, $d+1 \leq j \leq d + d_m$ from $\mu_k^{(g)}$ by $y_{ij,k}^{(g)}$, $1 \leq i \leq d$, $1 \leq j \leq d_m$ and the variables $x_{ij,k}^{(g)}$, $d+1 \leq i, j \leq d + d_m$ from $\eta_k^{(g)}$ by $z_{ij,k}^{(g)}$, $1 \leq i, j \leq d_m$.

Each $h_t = h_t(\omega_1^{(g_1)}, \ldots, \omega_{k_t}^{(g_t)})$ is a matrix of the form $(h_{rs,t})$, where each $h_{rs,t} = h_{rs,t}(x_{ij,k}^{(g)})$ is a polynomial in the variables $x_{ij,k}^{(g)}$, for $1 \leq i, j \leq d$ and $k \geq 1$. The matrix μ has the form (y_{pq}), $1 \leq p \leq d$ and $1 \leq q \leq d_m$.

With this notation, multiplying matrices we have

$$h_t \mu = \left(\sum_{l=1}^{d} h_{rl,t} y_{ls} \right), \quad 1 \leq r \leq d, \quad 1 \leq s \leq d_m$$

If $f_t = f_t(\eta_1^{(g_1)}, \ldots, \eta_{n_t}^{(g_t)})$, we can write it as a matrix of the form $(f_{pq,t})$, with $f_{pq,t}$ a polynomial in the variables $z_{ij,k}^{(g)}$, $1 \leq i, j \leq d_m$ and $k \geq 1$. Then for each t, we have

$$h_t \mu f_t = \left(\sum_{k=1}^{d_m} \sum_{l=1}^{d} h_{rl,t} y_{lk} f_{ks,t} \right), \quad 1 \leq r \leq d, \quad 1 \leq s \leq d_m.$$

And supposing $\sum_{t=1}^n h_t \mu g_t = 0$, we obtain

$$0 = \sum_{t=1}^{n} h_t \mu f_t = \left(\sum_{t=1}^{n} \sum_{k=1}^{d_m} \sum_{l=1}^{d} h_{rl,t} y_{lk} f_{ks,t} \right), \quad 1 \leq r \leq d, \quad 1 \leq s \leq d_m$$

Hence each entry of the above matrix is zero, i.e., for $1 \le r \le d$ and $1 \le s \le d_m$,

$$\sum_{t=1}^{n}\sum_{k=1}^{d_m}\sum_{l=1}^{d} h_{rl,t}y_{lk}f_{ks,t} = 0.$$

Since the polynomials $h_{lr,t}$ depend only on the variables x, and the polynomials $f_{ks,t}$ depend only on the variables z, analyzing the degree of the variables y_{kl} in the above sum, we have that for each k and l

$$\sum_{t=1}^{n} h_{rl,t}y_{lk}f_{ks,t} = 0.$$

For $r = s = l = 1$ we have the system of equations

$$\begin{cases} \sum_{t=1}^{n} h_{11,t}y_{11}f_{11,t} = 0 \\ \sum_{t=1}^{n} h_{11,t}y_{12}f_{21,t} = 0 \\ \qquad\vdots \\ \sum_{t=1}^{n} h_{11,t}y_{1d}f_{d1,t} = 0 \end{cases}$$

Since $\{f_1, \dots, f_n\}$ is linearly independent over F, Corollary 29 implies that the set

$$\left\{ \begin{pmatrix} f_{11,t} \\ \vdots \\ f_{d1,t} \end{pmatrix},\ 1 \le t \le n \right\} \subseteq M_{d\times 1}(U_{\mathbb{Z}_2}(E))$$

is also linearly independent over F.

If $U_{\mathbb{Z}_2}(E)$ has free generators $x_{ij,k}^{(g)}$, y_{ij}, $z_{ij,k}^{(g)}$, we can consider an order in this set of generators such that $x < y < z$. In Remark 3, we exhibit a basis to the relatively free $\mathbb{Z}_2$-graded algebra of E. One can easily see that such basis has the property that if two elements of this basis which depends on disjoint sets of variables, say X and Y, are multiplied, one gets another basis element. In fact such new basis element cannot be obtained by other product of basis elements depending on the same sets of variables X and Y. If such property holds, we say this basis is multiplicative.

We use such a basis to write $f_{11,t} = \sum \alpha_{I_{11,t}} b_{I_{11,t}}$, where $\alpha_{I_{11,t}} \in F$, and $b_{I_{11,t}}$ are basis elements of $U_{\mathbb{Z}_2}(E)$ which depends only on the variables z.

Also $h_{11,t} = \sum \beta_{J_{11,t}} c_{J_{11,t}}$, where $\beta_{I_{11,t}} \in F$ and $c_{J_{11,t}}$ are basis elements of $U_{\mathbb{Z}_2}$ which depends only on the variables x.

With this notation, the above equations become

$$\sum_{I}\sum_{J}\sum_{t=1}^{n} \alpha_{I_{11,t}}\beta_{J_{11,t}}b_{I_{11,t}}yc_{J_{11,t}} = 0$$

As we have seen the basis of $U_{\mathbb{Z}_2}(E)$ is multiplicative. Hence $\alpha_{I_{11,t}}\beta_{J_{11,t}} = 0$, for every t. In a similar way we show that $\alpha_{I_{k1,t}}\beta_{J_{11,t}} = 0$, for every $1 \le k \le d$. Then for each fixed t, we have $\alpha_{I_{k1,t}}\beta_{J_{11,t}} = 0$. If $\alpha_{I_{k1,t}} = 0$. for every k then $f_t = 0$, which is not true, since $\{f_1, \dots, f_n\}$ is linearly independent. Hence at least one of them is nonzero, and that implies $\beta_{J_{11,t}} = 0$. I.e., $f_{11,t} = 0$, for each t. The same arguments show that $f_{ij,t} = 0$, for each i and j, and this completes the proof. $\square$

8. Conclusions

We observe that the model we presented in this paper transfers the information about the graded identities of a graded algebra A into the ideal of graded plynomial identities of a block triangular matrix $UT(d_1, \ldots, d_m; A)$. Dealing with the proof of Theorem 30, we note that the main trick is the fact that the basis of the relatively free $\mathbb{Z}_2$-graded algebra of E, when naturally graded, is multiplicative. Moreover, we may observe that the map $\deg(\cdot)_\infty$ induces a grading over E such that its relatively free $\mathbb{Z}_2$-graded algebra has a multiplicative basis. We can state the following.

Theorem 31. *Let E be the Grassmann algebra endowed with the $\mathbb{Z}_2$-grading induced by the map $\deg(\cdot)_\infty$. If $d_1, \ldots, d_m$ are positive integers, then*

$$T_{\mathbb{Z}_2}(UT(d_1, \ldots, d_m; E)) = T_{\mathbb{Z}_2}(M_{d_1}(E)) \cdots T_{\mathbb{Z}_2}(M_{d_m}(E)).$$

As we already noted in Remark 2, the only homogeneous $\mathbb{Z}_2$-grading over E that gives rise to a monomial identity is the one induced by the map $\deg(\cdot)_{k^*}$. Indeed, the relatively free $\mathbb{Z}_2$-graded algebra has not a multiplicative basis. In fact, the Theorems 30 and 31 can not be generalized. We have the following.

Proposition 32. *Let R be the $\mathbb{Z}_2$-graded algebra*

$$R := \begin{pmatrix} E & E \\ 0 & E \end{pmatrix},$$

where the $\mathbb{Z}_2$-grading is induced by that of E. If E is graded by $\deg(\cdot_{k^*})$, then

$$T_{\mathbb{Z}_2}(E)T_{\mathbb{Z}_2}(E) \subset T_{\mathbb{Z}_2}(R).$$

Proof. It is easy to see that $z_1 \cdots z_{k+1}$ is a graded identity of R but it is clearly not a consequence of the product $T_{\mathbb{Z}_2}(E)T_{\mathbb{Z}_2}(E)$ because $T_{\mathbb{Z}_2}(E)$ contains the identity $z_1 \cdots z_{k+1}$ (see Theorem 1). □

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